Previously, we examined problems like

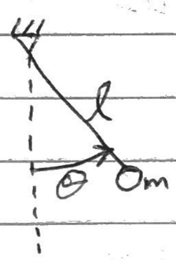
$$\frac{d\underline{x}}{dt} = f(t, \underline{x}, \underline{u})$$

and if $f(t, \underline{x}, \underline{u})$ was a non-linear function, we solved the differential equation numerically using Euler's method.

But, we only gain insight into problem after running the simulation. Need intuition for the system for design.

Build up intuition using <u>linearization</u>.

$\boxed{Ex}$



$$m\ell \frac{d^2\theta}{dt^2} = -mg\sin(\theta)$$

$$\boxed{\frac{d^2\theta}{dt^2} = \frac{-g}{\ell}\sin(\theta)}$$

<u>Step 1:</u> Find equilibrium points.

$\dot{\theta} = \ddot{\theta} = 0$

$$-\frac{g}{\ell}\sin(\theta) = 0$$

$$\sin(\theta) = 0$$

$$\theta = 0, \pi, 2\pi, \ldots$$

<u>Step 2:</u> Taylor expand right hand side around equilibrium

$$\theta = \theta_e + \Delta\theta$$

$$\sin(\theta_e + \Delta\theta) = \sin(\theta_e) + \cos(\theta_e)\Delta\theta - \frac{1}{2}\sin(\theta_e)\Delta\theta^2 - \frac{1}{6}\cos(\theta_e)\Delta\theta^3 + \ldots$$

<u>Step 3:</u> Assume $\Delta\theta$ is small so we only keep the linear terms

$$\sin(\theta_e + \Delta\theta) \approx \sin(\theta_e) + \cos(\theta_e)\Delta\theta$$

Step 4: Evaluate at Equilibrium

$\theta = \theta_e + \Delta\theta$

$\frac{d\theta}{dt} = \frac{d\Delta\theta}{dt}$ $\quad$ $\frac{d^2\theta}{dt^2} = \frac{d^2\Delta\theta}{dt^2}$

$\theta_e = 0, 2\pi, 4\pi, \ldots$

$$\frac{d^2\Delta\theta}{dt^2} = -\frac{g}{l}\left(\sin(\theta_e) + \cos(\theta_e)\Delta\theta\right)$$

$$= -\frac{g}{l}\left(0 + \Delta\theta\right)$$

$$\frac{d^2\Delta\theta}{dt^2} = -\frac{g}{l}\Delta\theta$$

$\theta_e = \pi, 3\pi, \ldots$

$$\frac{d^2\Delta\theta}{dt^2} = -\frac{g}{l}\left(\sin(\theta_e) + \cos(\theta_e)\Delta\theta\right)$$

$$= -\frac{g}{l}\left(0 - \Delta\theta\right)$$

$$\frac{d^2\Delta\theta}{dt^2} = \frac{g}{l}\Delta\theta$$

Step 5: Solve for $\Delta\theta$

$\Delta\theta(t=0) = \Delta\theta_0$ $\quad$ $\Delta\dot{\theta}(t=0) = 0$

$$\boxed{\begin{array}{l}\theta_e = 0, 2\pi, 4\pi, \ldots \\ \Delta\theta = \Delta\theta_0 \cos\left(t\sqrt{\frac{g}{l}}\right)\end{array}}$$

$$\boxed{\begin{array}{l}\theta_e = \pi, 3\pi, 5\pi, \ldots \\ \Delta\theta = \Delta\theta_0 \cosh\left(t\sqrt{\frac{g}{l}}\right)\end{array}}$$

In general:

$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, \underline{u})$$

Step 1: find equilibrium for given inputs $\hat{\underline{u}}$

$$0 = \underline{f}(\underline{x}_e, \hat{\underline{u}})$$

Step 2: Taylor series expand around $\underline{x_e}$ and $\hat{\underline{u}}$

$\underline{u} = \hat{\underline{u}} + \Delta\underline{u}$

$\underline{x} = \underline{x}_e + \Delta\underline{x}$

$$f_1(\underline{x}_e + \Delta\underline{x}, \hat{\underline{u}} + \Delta\underline{u}) = f_1(\underline{x}_e, \hat{\underline{u}}) + \frac{\partial f_1}{\partial x_1}(\underline{x}_e, \hat{\underline{u}})\Delta x_1 + \frac{\partial f_1}{\partial x_2}(\underline{x}_e, \hat{\underline{u}})\Delta x_2 + \dots$$
$$+ \frac{\partial f_1}{\partial u_1}(\underline{x}_e, \hat{\underline{u}})\Delta u_1 + \frac{\partial f_1}{\partial u_2}(\underline{x}_e, \hat{\underline{u}})\Delta u_2 + \dots$$

*Do the same for all the $f_i(\underline{x}_e + \Delta\underline{x}, \hat{\underline{u}} + \Delta\underline{u})$ to eventually get

$$\underline{f}(\underline{x}_e + \Delta\underline{x}, \hat{\underline{u}} + \Delta\underline{u}) \approx \cancelto{0}{\underline{f}(\underline{x}_e, \hat{\underline{u}})} + \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\underline{x}_e, \hat{\underline{u}}} \begin{Bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{Bmatrix}$$
$$+ \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & & \frac{\partial f_n}{\partial u_n} \end{bmatrix}\right|_{\underline{x}_e, \hat{\underline{u}}} \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_n \end{Bmatrix}$$

$$\underline{f}(\underline{x}_e + \Delta\underline{x}, \hat{\underline{u}} + \Delta\underline{u}) = A\Delta\underline{x} + B\Delta\underline{u}$$

* A and B are matrices that are evaluated at an equilibrium point

* Must evaluate each one for however many equilibrium points.

Linearized equations: $\boxed{\frac{d\Delta \underline{x}}{dt} = A\Delta\underline{x} + B\Delta\underline{u}}$

Ex $\frac{dx}{dt} = x(1-y)$

$\frac{dy}{dt} = y(x-1)$

Step 1: Equilibrium points

$0 = x(1-y)$ $\quad\quad$ $x=0$ $\quad\quad$ $\boxed{\begin{matrix} x=1 \\ y=1 \end{matrix}}$

$0 = y(x-1)$ $\quad\quad$ $\underbrace{y=0}_{\text{trivial, so ignore here}}$

Step 2: $A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \left.\begin{bmatrix} (1-y) & -x \\ y & (x-1) \end{bmatrix}\right|_{x=1, y=1}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\boxed{\frac{d}{dt}\begin{Bmatrix} \Delta x \\ \Delta y \end{Bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} \Delta x \\ \Delta y \end{Bmatrix}}$$

linear Solution:

$x(t) = 1 + \Delta x_0 \cos(t) - \Delta y_0 \sin(t)$

$y(t) = 1 + \Delta x_0 \sin(t) + \Delta y_0 \cos(t)$